Computational Aspects of Nonsimilar Solutions of the Rarefied Leading Edge on a Cone

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Theme

THE flowfield structure of the viscous hypersonic flow over an unyawed cone is examined in the context of an initial value problem. Only the merged layer regime on the cone is considered. The objectives of this research program have been to extend the two-layer formulation of Shorenstein and Probstein to nonsimilar viscous layer solutions, and to determine the effects of initial profiles on the flow. Attention is given to the details of the initial data profiles and the finite-difference form of the model equations. Physically acceptable and mathematically self-consistent solutions are obtained. This investigation has resulted in the following: a) The two-layer formulation has been extended to nonsimilar viscous flows; b) The development of the merged layer profile as a function of initial data has been determined; c) A novel implicit numerical scheme has been developed which gives admissible solutions.

Contents

The merged layer region has been studied by several investigators; a review of the results of these investigations is given by the authors.[1] In the merged layer regime, any physical model must contain the essential features of this regime. These features are that the shock thickness and viscous layer thickness are of the same order and the absence of an inviscid region. Thus, the merged layer region may be modeled as consisting of two layers.[2]

Previous investigators employing the two-layer approach have assumed local similarity in the viscous layer and that at the shock-viscous layer interface the streamwise pressure gradient can be obtained using Euler's equations. This approach is incorrect in the merged regime because of the absence of an inviscid region between the non Rankine-Hugoniot shock and the viscous layer, and because the shear at the interface has a finite value. Since a non Rankine-Hugoniot shock envelopes the viscous layer at the outer edge, we are required to solve the shock structure problem in order to specify boundary conditions at the outer edge of the viscous layer. These boundary conditions include matching of state variables and velocity components at the interface. The conical shock structure is treated within the Navier-Stokes formulation. Assuming no gradients in the azimuthal direction, the transverse curvature of the shock can be included to all orders of inverse Reynolds number based on the local shock radius.

The viscous layer adjacent to the surface of the cone is taken to be nonsimilar and of the boundary-layer type. The viscous layer is modeled after that of Yasuhara with modified external pressure field and boundary conditions. The original freestream boundary conditions of Yasuhara and the classical Rankine-Hugoniot conditions downstream of the conical shock are replaced by matching the mass flux, the tangential shear, velocity components, and state variables at the shock wave-viscous layer interface.

Attention is given to the details of the initial data profiles. We also note the finite-difference form of the model equations which are required for the generation of a solution which is physically acceptable and mathematically self-consistent. Heat transfer rate and viscous shear at the surface of the cone are computed. First-order corrections for shock curvature, velocity slip, and temperature jump at the cone surface are not considered in this paper. The proposed method does provide a straightforward means for including these effects.[1]

The following relations can be obtained by integrating the zeroth order conservation equations for the shock structure and applying the freestream boundary conditions:

$$\rho^o v^o = -\sin\theta_s \tag{1}$$

$$p_s^o = \frac{\gamma-1}{\gamma}\,\rho_s^o\left(g_s^* - \frac{1}{2}A_s^2\right) \tag{2}$$

$$\sin^2\theta_s = \frac{\rho_s^o(p_s^o - 1/\gamma M_\infty^2)}{(\rho_s^o - 1)} \tag{3}$$

$$\left[\frac{4}{3}\mu^o\frac{\partial v^o}{\partial\eta}\right]_{\eta=0} = -(p_\infty + \sin^2\theta_s) + (p_s^o - v_s^o\sin\theta_s) \tag{4}$$

$$\left[\mu^o\frac{\partial u^o}{\partial\eta}\right]_{\eta=0} = \sin\theta_s\cos\theta_s - u_s^o\sin\theta_s \tag{5}$$

$$h_s^o = -\sin\theta_s + \frac{\sin\theta_s}{p_s^o} + \frac{1}{\gamma M_\infty^2} + \frac{3}{2}u_s^{o\,2} - u_s^o\cos\theta_s - \frac{1}{2}\cos^2\theta_s \tag{6}$$

where ρ, p, h, u, and v are the density, pressure, enthalpy, normal velocity, and tangential velocity components, respectively. μ, γ, θ, and M are the coefficient of shear viscosity, ratio of specific heats, angle measured relative to the flow direction, and flow Mach number. The subscripts s and ∞ refer to the variables at the downstream edge of the shock and in the freestream, respectively. Also,

$$g_s^* = H_s^*/u_\infty^{*\,2}, \quad A_s = u_s^*/u_\infty^*, \quad \eta = \rho_\infty u_\infty^* y/\mu_\infty \tag{7}$$

where H denotes the total enthalpy, and the superscripts 0 and * refer to the zeroth-order variables and to the viscous layer variables, respectively. Equations (1-7) provide expressions for the flow variables at the viscous layer shock wave interface. They also provide a means of computing the transformed independent variables in the shock layer and in the viscous layer in terms of spatial coordinates.

The equations of motion governing the viscous layer are nonlinear third-order partial differential equations. For a Prandtl number of unity, these equations are:

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$$N^*R \frac{\partial^3 f^*}{\partial \eta^{*3}} + [f^* + 2\xi^* \frac{\partial f^*}{\partial \xi^*} + \frac{\partial}{\partial \eta^*} \{\ell n(r^*\mu^* u_s^*)\}] \times \frac{\partial^2 f^*}{\partial \eta^{*2}} - 2\xi^* \frac{\partial f^*}{\partial \eta^*} \frac{\partial 2_f}{\partial \xi^* \partial \eta^*} = \frac{2\xi^*}{\rho^* u_s^{*2}} \frac{\partial p^*}{\partial \xi^*} \quad (8)$$

$$\frac{\partial p^*}{\partial \eta^*} = 0 \quad (9)$$

$$N^*R \frac{\partial^2 g^*}{\partial \eta^{*2}} + [N^*R \frac{\partial}{\partial \eta^*} \{\ln(\mu^* r^*)\} + (f^* + 2\xi^* \frac{\partial f^*}{\partial \xi^*})] \frac{\partial g^*}{\partial \eta^*} - 2\xi^* \frac{\partial f^*}{\partial \eta^*} \frac{\partial g^*}{\partial \xi^*} = 0 \quad (10)$$

where $N^* = \rho^*\mu^*/(\rho^*\mu^*)_w$, $R = r^*/r_w^*$ and the subscript w denote variables at the cone's surface. The transformed variables are

$$\xi^* = \int_0^{x^*} (\rho^*\mu^*)_w u_s^* r_w^2 dx^*, \quad \eta^* = \frac{u_s^*}{(2\xi^*)^{1/2}} \int_0^{y^*} \rho^* r^* dy^*,$$

$$f^* = \int_0^{\eta^*} \frac{\eta^*}{u_s^*} d\eta^* \quad (11)$$

where r is the radial coordinate.

For the purpose of formulating an implicit finite-difference approximation for Eqs. (8-11), we divide the η^* interval into $(M-1)$ parts so that there is a total of M points from $\eta^* = 0$ to $\eta^* = \eta_s^*$. This involves M unknowns. We have 2 conditions imposed at the wall and 1 at the interface. We employ an $0(h^2)$ central difference scheme from the 2nd to the $(M-2)$th point. We apply an $0(h^2)$ backward difference scheme at the $(M-1)$th point. The total number of equations thus obtained is $(M-3)$ involving M unknowns. Three more equations are obtained by putting the boundary conditions in finite-difference form. The implicit finite-difference equation resulting from an application of $0(h^2)$ central differences to Eq. (8) is the following

$$f^*_{i+1,i+2} + f^*_{i+1,j+1}[-2 + \frac{2h\delta}{R} - \frac{AK}{R} h^2 Fl] + f^*_{i+1,j}[-4 \frac{h\delta}{R}] + f^*_{i+1,j-1}[2 + \frac{2h\delta}{R} + \frac{AK}{R} h^2 Fl] + f^*_{i+1,j-2} = \frac{\partial p^*}{\partial \xi^*} \frac{2\xi^*}{\rho^* u_s^{*2}} \frac{2h^3}{R} + \frac{AK}{R} h^2 Fl \; [f^*_{i-1,j-1} + f^*_{i-1,j+1}] \quad (12)$$

where $AK = \xi^*/\Delta\xi^*$, $h = \Delta\eta^*$, $Fl = (\partial f^*/\partial \eta^*)_{i,j}$

$$\delta = f^*_{i,j} + 2AK(f^*_{i,j} - f^*_{i-j}) + \cos\theta_c (2\xi^*)^{1/2} / r_w^2 R u_s^* \rho^*$$

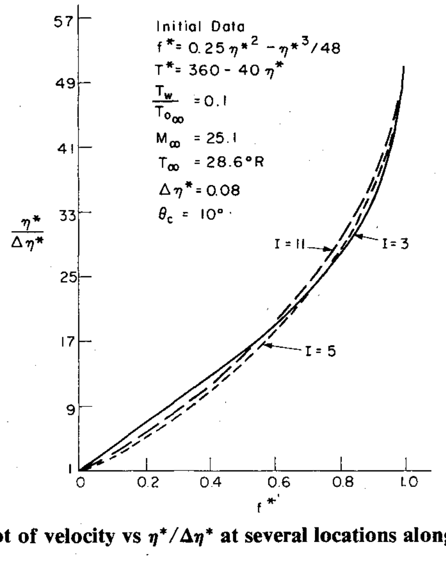

Fig. 1 Plot of velocity vs $\eta^*/\Delta\eta^*$ at several locations along the cone surface.

Table 1 Computed values at cone surface for initial data defined in Fig. 1

I	V (Rarefaction parameter	$f^{*\prime\prime}(0)$ Shear	$(1/T_w^*)(\partial T^*/\partial \eta^*)\eta^* = 0$ Heat transfer rate
3	0.2087	0.445	1.856
4	0.1917	0.734	6.042
5	0.1812	0.927	9.886
6	0.1738	0.776	10.836
7	0.1681	0.760	10.858
8	0.1635	0.746	10.975
9	0.1596	0.753	11.098
10	0.1563	0.753	11.265
11	0.1535	0.757	11.413
12	0.1509	0.756	11.546
13	0.1486	0.757	11.676

Similarly, an $0(h^2)$ backward difference form of Eq. (8) may be obtained. The finite-difference equations for Eq. (8) along with the boundary conditions in finite-difference form give M equations in M unknowns. The $[M \times M]$ matrix for the coefficients of $f^*_{i,j}$'s is a hepta diagonal matrix with 2 upper co-diagonals and 4 lower co-diagonals. The finite-difference equations are linear algebraic equations, since known values of $f^*_{i,j}$ are used as coefficients for the term $f^*(\partial^2 f^*/\partial \eta^{*2})$. The finite-difference equations for Eq. (10) give M equations in M unknowns. For the coefficients of g, we get an $[M \times M]$ tridiagonal matrix with 1 upper and 1 lower co-diagonal.

Equations (8-10) are parabolic. Hence, boundary conditions and initial conditions are required in order to obtain a solution. To specify the initial conditions, various types of initial profiles are employed and a selection is made which is based on meaningful asymptotic profiles. The conditions at the interface are obtained by matching the shock profile with the viscous layer profile. The band structured matrix equations for the coefficients of $f^*_{i,j}$ and $g^*_{i,j}$ are solved using a Gauss elimination procedure.

The relative effects of initial profiles are summarized by the authors.[1] It is noted that if $f^{*\prime\prime}(\xi_i^*, 0)$ is larger some value (which is approximately 0.5), then the $f^{*\prime\prime}(\xi,0)$ curve is monotonically decreasing, and if $f^{*\prime\prime}(\xi_i^*, 0)$ is smaller than that value, the $f^{*\prime\prime}(\xi^*,0)$ curve has a maximum at some point. We also note that the larger the value of $f^{*\prime\prime}(\xi_i^*, \eta_i^*)$, the smaller is the asymptotic value of $f^{*\prime\prime}(\xi^*,0)$.

Figure 1 is a plot of $f^{*\prime}$ at different locations along the surface of the cone. The $f^{*\prime}$ profiles remain almost unchanged as we move downstream. There is a small change in $f^{*\prime\prime}$ at the wall and at the interface as we move downstream and $f^{*\prime\prime}$ takes on an approximately constant value at the wall after a short distance. Table 1 contains values of the viscous shear, $f^{*\prime\prime}(\xi^*,0)$, and the heat transfer rate at the surface of the cone. These results are concomitant with the initial data defined in Fig. 1. The results in Table 1 indicate that $f^{*\prime\prime}(\xi^*, 0)$ is not monotonic as we approach the leading edge from the downstream direction. This result is attributed to numerical instabilities. The heat transfer rate is monotonic.

In conclusion, a novel implicit finite-difference scheme has been developed to generate solutions per the two-layer formulation of the merged regime on a cone in hypersonic flow. The relative effects of initial profiles on the nonsimilar viscous layer model have been obtained.

References

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